

ADDENDUM: VIRTUALLY FREE PRO- p GROUPS WHOSE TORSION ELEMENTS HAVE FINITE CENTRALIZER

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ABSTRACT. We fill details in the proof of [HZ, Lemma 13] (that is [4, Lemma 3.2]). For easier reading we include the relevant part of section 3 ibidem.

20E18 (primary), 20E06, 22C05 (secondary).

3. HNN-EMBEDDING

We introduce a notion of a pro- p HNN-group as a generalization of pro- p HNN-extension in the sense of [1, page 97]. It also can be defined as a sequence of pro- p HNN-extensions. During the definition to follow, i belongs to a finite set I of indices.

Definition 1. Let G be a pro- p group and A_i, B_i be subgroups of G with isomorphisms $\phi_i : A_i \rightarrow B_i$. The pro- p HNN-group is then a pro- p group $\text{HNN}(G, A_i, \phi_i, z_i)$ having presentation

$$\text{HNN}(G, A_i, \phi_i, z_i) = \langle G, z_i \mid \text{rel}(G), \forall a_i \in A_i : a_i^{z_i} = \phi_i(a_i) \rangle.$$

The group G is called the *base group*, A_i, B_i are called *associated subgroups* and z_i are called the *stable letters*.

For the rest of this section let G be a finitely generated virtually free pro- p group, and fix an open free pro- p normal subgroup F of G of minimal index. Also suppose that $C_F(t) = \{1\}$ for every torsion element $t \in G$. Let $K := G/F$ and form $G_0 := G \amalg K$. Let $\psi : G \rightarrow K$ denote the canonical projection. It extends to an epimorphism $\psi_0 : G_0 \rightarrow K$, by sending $g \in G$ to $gF/F \in K$ and each $k \in K$ identically to k , and using the universal property of the free pro- p product. Remark that the kernel of ψ_0 , say L , is an open subgroup of G_0 and, as $L \cap G = F$ and $L \cap K = \{1\}$, as a consequence of the pro- p version of the Kurosh subgroup theorem, [2, Theorem 9.1.9], L is free pro- p . Let I be the set of all G -conjugacy classes of maximal finite subgroups of G and observe that in light of [HZ, Lemma 8] the set I is finite. Fix, for every $i \in I$, a finite subgroup K_i of G in the G -conjugacy class i . We define a pro- p HNN-group by considering first $\tilde{G}_0 := G_0 \amalg F(z_i \mid i \in I)$ with z_i constituting a free set of generators, and then taking the normal subgroup R in \tilde{G}_0 generated by all elements of the form $k_i^{z_i} \psi(k_i)^{-1}$, with $k_i \in K_i$ and $i \in I$. Finally set

$$\tilde{G} := \tilde{G}_0/R,$$

and, since all K_i are finite, by [2, Prop. 9.4.3] it is a proper HNN-group $\text{HNN}(G_0, K_i, \phi_i, z_i)$, where $\phi_i := \psi|_{K_i}$, G_0 is the base group, the K_i are associated subgroups, and the z_i form a set of stable letters in the sense of Definition 1.

Let us show that \tilde{G} is virtually free pro- p . The above epimorphism $\psi_0 : G_0 \rightarrow K$ extends to $\tilde{G} \rightarrow K$ by the universal property of the HNN-extension, so \tilde{G} is a semidirect product $\tilde{F} \rtimes K$ of its kernel \tilde{F} with K . By [3, Lemma 10], every open torsion-free subgroup of \tilde{G} must be free pro- p , so \tilde{F} is free pro- p .

The objective of the section is to give a more detailed version of the proof of [HZ, Lemma 13], i.e., to show that the centralizers of torsion elements in \tilde{G} are finite.

Lemma 2. *Let $\tilde{G} = \text{HNN}(G_0, K_i, \phi_i, z_i)$ and \tilde{F} be as explained. Then $C_{\tilde{F}}(t) = 1$ for every torsion element $t \in \tilde{G}$.*

Proof. There is a standard pro- p tree $S := S(\tilde{G})$ associated to $\tilde{G} := \text{HNN}(G_0, K_i, \phi_i, z_i)$ on which \tilde{G} acts naturally such that the vertex stabilizers are conjugates of G_0 and each edge stabilizer is a conjugate of some K_i .

Claim: Let e_1, e_2 be two edges of S with a common vertex v which is not terminal vertex of both of them. Then the intersection of the stabilizers $\tilde{G}_{e_1} \cap \tilde{G}_{e_2}$ is trivial.

Proof of the Claim: By translating e_1, e_2, v if necessary we may assume that G_0 is the stabilizer of v . Then we have two cases:

1) v is initial vertex of e_1 and e_2 . Then $\tilde{G}_{e_1} = K_i^g$ and $\tilde{G}_{e_2} = K_{i'}^{g'}$ with $g, g' \in G_0$ and either $i \neq i'$ or $g \notin K_i g'$ and by construction of \tilde{G} we have $K_i^g \neq K_{i'}^{g'}$ if $t \neq t'$. Suppose that $K_i^g \cap K_{i'}^{g'} \neq \{1\}$. Then, since $G_0 = G \amalg K$, we may apply [HZ, Theorem 2.9], in order to deduce the existence of $g_0 \in G_0$ with $K_i^{gg_0} \cap K_{i'}^{g'g_0} \leq G$. Now by [3, Lemma 2.7] two distinct maximal finite subgroups of G_0 have trivial intersection. So we have $K_i^g \cap K_{i'}^{g'} = \{1\}$, as needed.

2) v is the terminal vertex of e_1 and the initial vertex of e_2 . Then $\tilde{G}_{e_1} = K^g$ and $\tilde{G}_{e_2} = K_{i'}^{g'}$ for $g, g' \in G_0$ so they intersect trivially by the definition of G_0 and [HZ, Theorem 2.9]. So the Claim holds.

Now pick a torsion element $t \in \tilde{G}$ and $\tilde{f} \in \tilde{F}$ with $t\tilde{f} = t$. Let $e \in E(S)$ be an edge stabilized by t . Then $\tilde{f}e$ is also stabilized by t and, as by [1, Theorem 3.7], the fixed set S^t is a subtree, the path $[e, \tilde{f}e]$ is fixed by t as well. Note that e and $\tilde{f}e$ cannot have a common vertex, since \tilde{f} cannot stabilize any vertex. Moreover, S^t is infinite since $\langle \tilde{f} \rangle$ is torsion free and acts freely on S^t . Now S^t is connected and Corollary 4 in the Appendix implies that it must have path components of arbitrary cardinality. Therefore we can choose e and \tilde{f} such that $[e, \tilde{f}e]$ contains at least 3 pairwise adjacent edges and so $[e, \tilde{f}e]$ contains at least one vertex which is not the terminal point of all its incident edges. Then by the Claim $t = 1$. \square

APPENDIX: PATH COMPONENTS OF FINITE DIAMETER IN A PROFINITE GRAPH

We shall need a general result about *profinite graphs*. Composition RS of binary relations R and S on a set X is defined as $xRSy$ if and only if there is $z \in X$ so that $(x, z) \in R$ and $(z, y) \in S$ holds. Define inductively $R^1 := R$ and $R^{n+1} := R^n R$ for $n \in \mathbb{N}$. Let R° denote the *converse* relation, i.e., $(x, y) \in R^\circ$ if and only if $(y, x) \in R$ and, as common, $\Delta := \{(x, x) \mid x \in X\}$ is the *diagonal*.

For an abstract graph Γ consider $R_0 := \{(x, y) \in \Gamma \times \Gamma \mid d_1(x) = d_0(y)\}$, and, set $R := R_0 \cup R_0^\circ \cup \Delta$. Then $xR^n y$ if and only if there is a geodesic of length not exceeding n in Γ containing x and y . The path-components of Γ turn out to be

the equivalence classes of $\Sigma := \bigcup_n R^n$. Define $\delta(\Gamma)$ to be the supremum of the diameters of its connected components then

$$\delta(\Gamma) \leq n \text{ if and only if } \Sigma = R^n.$$

When X is a compact space then a standard compactness argument implies that with R compact every R^n is compact. Every profinite graph is an abstract one.

Lemma 3. *Let Γ be a profinite graph and $\delta(\Gamma) < \infty$. Then the path components of Γ are exactly the connected components.*

Proof. Set $n := \delta(\Gamma)$. Since R is closed $\Sigma = R^n$ is a closed equivalence relation. Hence its equivalence classes, the path components, are all closed. The quotient graph Γ/Σ does not contain edges and so it is totally disconnected. Since every connected component of $x \in \Gamma$ contains the path component of x , each path component is a connected component of Γ . \square

Corollary 4. *A connected profinite graph Γ with $\delta(\Gamma) < \infty$ consists of a single path component.*

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